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Log-Concavity and the Exponential Formula

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A 1996 result of Bender and Canfield showed that passing a log-concave sequence through the exponential formula resulted in a log-concave sequence which was almost log-convex. We generalize that result to q-log-concavity. Our proof follows Bender

from the first part. We also give several corollaries and examples.

1. INTRODUCTION

Many combinatorial sequences display a certain "normal" structural behavior. One important manifestation of this behavior is log-concavity [3, 14]. In recent years there has been a growing interest in the q–log-concavity of q–analogs of combinatorial sequences [4–6, 9, 10].

In [2] Bender and Canfield introduced a method to generate a new sequence, which is log-concave and almost log-convex, from an old log-concave sequence. In this paper we generalize this result by observing that the almost log-convex conclusion follows from the log-concave part, without using the substitutions that they introduced. Our proof is based on symmetric functions. This allows us to apply the theorem to sequences of polynomials and derive q–analogs of Bender and Canfield's results. We obtain further examples by noting that if the original sequence is the sum of q–log-concave sequences, then the output sequence is q–log-concave and almost q–log-convex.

2. RESULTS

THEOREM 1. *Let $p_0 = 1$ and let $p_1, p_2, \ldots$ be indeterminates. Let*

$$\mathscr{Y} = \{p_1, p_2, \ldots\} \cup \{p_j\, p_k - p_{j-1}\, p_{k+1} : 0 < j \leqslant k\}$$

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and let

$$\sum_{n\geqslant 0} h_n u^n = \sum_{n\geqslant 0} \frac{g_n u^n}{n!} = \exp\left(\sum_{j\geqslant 1} \frac{p_j u^j}{j}\right). \tag{1}$$

Then for $1\leqslant m\leqslant n$,

(i) $(n+1)g_m g_n - mg_{m-1} g_{n+1}$

is a polynomial in $\mathscr{Y}$ *with non-negative integer coefficients and*

(ii) $g_{m-1} g_{n+1} - g_m g_n$

is also a polynomial in $\mathscr{Y}$ *with non-negative integer coefficients.*

Since $n!\, h_n = g_n$ the differences (i) and (ii) are equivalent to

$$m!\, n![h_m h_n - h_{m-1} h_{n+1}] \qquad \text{and} \tag{2}$$

$$(m-1)!\, n![(n+1)\, h_{m-1} h_{n+1} - mh_m h_n], \tag{3}$$

respectively. Also g_n is a polynomial in $p_1, ..., p_n$ with integer coefficients [8, I (2.14′)].

Part (i) of Theorem 1 is a result of Bender and Canfield and for the proof we refer the reader to their exposition in [2]. Part (ii) is a generalization of a theorem of Bender and Canfield [2]. Our proof, based on the theory of symmetric functions, is different from their combinatorial proof. Part (ii) follows from (i) via (2), (3), and the following lemma.

LEMMA 2. *Let* $p_1, p_2, p_3, ...$ *and* $h_0, h_1, h_2, ...$ *be sequences related by the identity*

$$\sum_{n\geqslant 0} h_n u^n = \exp\left(\sum_{j\geqslant 1} p_j \frac{u^j}{j}\right)$$

and let

$$\mathscr{X} = \{p_1, p_2, ...\} \cup \{h_1, h_2, ...\} \cup \{h_j h_k - h_{j-1} h_{k+1} : 0 < j \leqslant k\}.$$

Then the difference

$$(n+1)\, h_{m-1} h_{n+1} - mh_m h_n \tag{4}$$

may be written as a polynomial in $\mathscr{X}$ *with non-negative integer coefficients.*

Proof. Using the identity $nh_n = \sum_{r=1}^{n} p_r h_{n-r}$ [8, I (2.11)] we can rewrite (4) as

$$h_{m-1}\left(\sum_{r=1}^{n+1} p_r\, h_{n+1-r}\right) - \left(\sum_{r=1}^{m} p_r\, h_{m-r}\right) h_n$$

and by collecting terms together we have

$$h_{m-1} \sum_{r=m+1}^{n+1} p_r\, h_{n+1-r} + \sum_{r=2}^{m} p_r [h_{m-1}\, h_{n+1-r} - h_{m-r}\, h_n].$$

Observe that the difference inside the square brackets is the sum of $(r-1)$-terms of the form $h_j\, h_k - h_{j-1} h_{k+1}$. ▮

THEOREM 3. *Let w be an integer greater than one. For $i = 1, 2, ..., w$ let $p_{i,0} = 1$ and let $p_{i,1}, p_{i,2}, ...$ be indeterminates. Let*

$$\mathscr{Y}_i = \{p_{i,1}, p_{i,2}, ...\} \cup \{p_{i,j}\, p_{i,k} - p_{i,j-1}\, p_{i,k+1} : 0 < j \leqslant k\}$$

and let

$$\sum_{n \geqslant 0} h_n\, u^n = \sum_{n \geqslant 0} \frac{g_n u^n}{n!} = \exp\left(\sum_{j \geqslant 1} \frac{(\sum_{i=1}^{w} p_{i,j})\, u^j}{j}\right).$$

Then for $1 \leqslant m \leqslant n$ both differences

$$h_m\, h_n - h_{m-1}\, h_{n+1} \tag{5}$$

$$(n+1)\, h_{m-1}\, h_{n+1} - m\, h_m h_n \tag{6}$$

are polynomials in $\bigcup_{i=1}^{w} \mathscr{Y}_i$ with non-negative rational coefficients.

Proof. Observe that the h's are the convolution of w sequences $h_{i,n}$ and from (2) it follows that $h_{i,m} h_{i,n} - h_{i,m-1} h_{i,n+1}$ is a polynomial in $\mathscr{Y}_i$ with non-negative rational coefficients. The Cauchy–Binet formula [7, p. 1], which expresses a minor of a product of two matrices as the sum of the product of certain minors of these matrices, implies that (5) is a polynomial in $\bigcup_{i=1}^{w} \mathscr{Y}_i$ with non-negative rational coefficients. That (6) is also a polynomial in $\bigcup_{i=1}^{w} \mathscr{Y}_i$ with non-negative rational coefficients follows from Lemma 2. ▮

3. APPLICATIONS

Let $h_k(w)$ be the *complete homogeneous symmetric function* in the variables $x_1, x_2, ..., x_w$ (see [8, p. 21]).

Corollary 4. *For $1 \leqslant m \leqslant n$ both differences*

$$h_m(w)\, h_n(w) - h_{m-1}(w)\, h_{n+1}(w) \tag{7}$$

$$(n+1)\, h_{m-1}(w)\, h_{n+1}(w) - m h_m(w)\, h_n(w) \tag{8}$$

may be written in terms of $\bigcup_{i=1}^{w} \mathscr{Y}_i$ with non-negative rational coefficients.

Proof. Let $p_{i,j} = x_i^j$ and apply Theorem 3. ▮

Remark 5. The differences (7) and (8) when written *in terms of the underlying variables* $x_1, ..., x_w$ are non-negative *integer* linear combinations of monomials in the x's. This follows by noting that (7) is the Schur function $s_{(n,m)}(x_1, ..., x_w)$ and (8) may be written as a linear combination of p's and skew Schur functions.

Our next application involves the q–analog of the binomial coefficients. These polynomials may be defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} \qquad \text{for } 0 \leqslant k \leqslant n, \tag{9}$$

where $[j]_q = 1 + q + q^2 + \cdots + q^{j-1}$ is the q-analog of the positive integer j and $[j]_q! = [1]_q\,[2]_q \cdots [j]_q$ is the q-version of $j!$. Although this definition is in terms of a rational function, $[{n \atop k}]_q$ is indeed a polynomial function. The interested reader can find more information on these polynomials in [1].

Corollary 6. *For $k \geqslant 0$ and $1 \leqslant m \leqslant n$ both differences*

$$\begin{bmatrix} m+k \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q - \begin{bmatrix} m-1+k \\ k \end{bmatrix}_q \begin{bmatrix} n+1+k \\ k \end{bmatrix}_q \tag{10}$$

$$(n+1) \begin{bmatrix} m-1+k \\ k \end{bmatrix}_q \begin{bmatrix} n+1+k \\ k \end{bmatrix}_q - m \begin{bmatrix} m+k \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \tag{11}$$

are polynomials in q with non-negative integer coefficients.

First Proof. Use Corollary 4 and Remark 5 with $w = k+1$ and $x_i = q^{i-1}$. The substitution $x_i = q^{i-1}$ is the well known *principal specialization* in the theory of symmetric functions [8, Ex. 3, p. 26]. ▮

Second Proof. For $j \geqslant 1$ let $p_j = 1 + q^j + q^{2j} + \cdots + q^{kj}$. Observe that $\exp\left(\sum_{j \geqslant 1} p_j u^j / j\right)$ is equal to $\prod_{i=0}^{k} (1 - q^i u)^{-1}$. By the q-binomial theorem [1, p. 36 (3.3.7)] the coefficient of u^n in the last product is $[{n+k \atop k}]_q$. It is well known [9, 10] that (10) is a polynomial in q with non-negative integer coefficients. Now apply Lemma 2 to show that (11) is also a polynomial in q with non-negative integer coefficients. ▮

Note that for $k=1$ the above corollary reduces to the q–analog of the positive integers.

Let $B^c_{n,k}$ be the collection of all set partitions of $\{1, 2, ..., n\}$ where each block has less than k elements and every block can be colored. Let

$$B^c_{n,k}(q) = \sum_{\tau \in B^c_{n,k}} q^{\#\text{ of colored blocks in } \tau}.$$

COROLLARY 7. *For $k>1$ and $1 \leqslant m \leqslant n$ both differences*

$$(n+1)\, B^c_{m,k}(q)\, B^c_{n,k}(q) - mB^c_{m-1,k}(q)\, B^c_{n+1,k}(q) \tag{12}$$

$$B^c_{m-1,k}(q)\, B^c_{n+1,k}(q) - B^c_{m,k}(q)\, B^c_{n,k}(q) \tag{13}$$

are polynomials in q with non-negative integer coefficients.

Proof. Let $p_0=1$, for $1 \leqslant j<k$ let $p_j=(1+q)/(j-1)!$, and for $j \geqslant k$ let $p_j=0$. Now apply Theorem 1. ▮

The constant term of the polynomial $B^c_{n,k}(q)$ is the number $B_{n,k}$ of Corollary 1.2 in [2].

Remark 8. One can interpret $B^c_{n,k}(q)$ in two ways. First, q is an indeterminate and so the coefficient of q^i counts the number of set partitions in $B^c_{n,k}$ where i blocks have been colored. Second, q is a *positive integer* and so $B^c_{n,k}(q)$ is the number of ways of constructing set partitions where each block can be given one of $q+1$ distinct colors.

Just as we have colored set partitions we can create *colored permutations* and their associated polynomials $\pi^c_{n,k}(q)$. These objects are permutations where each cycle can be colored. In this case let $p_0=1$, for $1 \leqslant j<k$ let $p_j=1+q$, and for $j \geqslant k$ set $p_j=0$. The constant term of the polynomial $\pi^c_{n,k}(q)$ is equal to the number $\pi_{n,k}$ of Corollary 1.1 in [2].

The next application involves *marked permutations*. Let S^{mk}_n be the set of all permutations in S_n with the property that each cycle of length j has a *marker* from the set $\{0, 1, 2, ..., j\}$. The weight of an element of S^{mk}_n is equal to the sum of its markers. Define the polynomials $\pi^{mk}_n(q)$ by

$$\pi^{mk}_n(q) = \sum_{\sigma \in S^{mk}_n} q^{\mathrm{wt}(\sigma)}. \tag{14}$$

For example, the set S_2^{mk} consists of the elements

$$(1)_0(2)_0, \quad (1)_1(2)_0, \quad (1)_0(2)_1, \quad (1)_1(2)_1, \quad (12)_0, \quad (12)_1, \quad (12)_2,$$

and so $\pi_2^{mk}(q)=2+3q+2q^2$.

The sequence $\pi_0^{mk}(1)$, $\pi_1^{mk}(1)$, $\pi_2^{mk}(1)$, ... is entry M1795 in [12].

COROLLARY 9. *For $1 \leqslant m \leqslant n$ both differences*

$$(n+1)\,\pi_m^{mk}(q)\,\pi_n^{mk}(q)-m\,\pi_{m-1}^{mk}(q)\,\pi_{n+1}^{mk}(q) \tag{15}$$

$$\pi_{m-1}^{mk}(q)\,\pi_{n+1}^{mk}(q)-\pi_m^{mk}(q)\,\pi_n^{mk}(q) \tag{16}$$

are polynomials in q with non-negative integer coefficients.

Proof. Let $p_j=[j+1]_q$ and apply Theorem 1. ∎

By restricting $p_j=[j+1]_q$ for $0 \leqslant j<k$ we obtain marked permutations with cycles of length at most $k-1$.

Each polynomial $\pi_n^{mk}(q)$ has degree n and is symmetric and unimodal. For if we fix the underlying permutation and let its markers vary we obtain the rank generating function [13, p. 99] for a product of chains. Also these polynomials satisfy the three term recurrence relation

$$\pi_n^{mk}(q)=(1+q)\,n\,\pi_{n-1}^{mk}(q)-q(n-1)^2\,\pi_{n-2}^{mk}(q) \tag{17}$$

with $\pi_0^{mk}(q)=1$ and $\pi_1^{mk}(q)=1+q$, which is a q–analog of the recurrence relation in [12, M1795].

To obtain *marked set partitions* we would choose $p_j=[j+1]_q/(j-1)!$.

4. REMARKS

Observe that (7) is the determinant of the Jacobi-Trudi matrix

$$\begin{pmatrix} h_n(w) & h_{n+1}(w) \\ h_{m-1}(w) & h_m(w) \end{pmatrix}$$

and by Theorem 1 if the p's form a Pólya frequency sequence of order 2 (see [7]), PF_2, then the h's will also be PF_2. A sequence $a_0, a_1, a_2, \ldots$ is PF_n if all the minors up to size n of the infinite matrix $(a_{j-i})_{1 \leqslant i,\, j \leqslant \infty}$, where $a_k=0$ if $k<0$, are non-negative. Unfortunately, the construction of Theorem 1 will not in general produce a PF_3 sequence. In fact, even if the

p_j's form a PF_∞ sequence, then the h_n's need not be PF_3. For example, take $p_j = j+1$. Then the determinant of the Jacobi-Trudi matrix for the partition 1^3 is equal to $-1/3$.

Also note that (8) can be expressed (up to sign) as the determinant of the matrix

$$\begin{pmatrix} h_n(w) & (n+1)\, h_{n+1}(w) \\ h_{m-1}(w) & m h_m(w) \end{pmatrix}.$$

We generalize this matrix as follows: for any partition λ and a column index k let $\mathrm{JT}(\lambda; k)$ denote the Jacobi-Trudi matrix of λ where each entry in the k-th column appears with a multiplicity equal to its subscript. For example, if $\lambda = (7, 4, 3)$, then

$$\mathrm{JT}(\lambda; 1) = \begin{pmatrix} 7h_7 & h_8 & h_9 \\ 3h_3 & h_4 & h_5 \\ 1h_1 & h_2 & h_3 \end{pmatrix} \quad \text{and} \quad \mathrm{JT}(\lambda; 3) = \begin{pmatrix} h_7 & h_8 & 9h_9 \\ h_3 & h_4 & 5h_5 \\ h_1 & h_2 & 3h_3 \end{pmatrix}.$$

The following theorem generalizes the result that (8) is monomial positive in the x_i's [11, Theorem 41].

THEOREM 10. *Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a partition and let $1 \leqslant k \leqslant n$. For $i > k$ let $\mu_i = (i-k, 1, ..., 1, 0, ..., 0)$ where there are $k-1$ ones and $n-k$ zeros. Then*

$$\det \mathrm{JT}(\lambda; k) = (-1)^{k-1} \sum_{i=k+1}^{\lambda_1+k} p_i\, s_{\lambda/\mu_i}.$$

Proof. Expand every element of the k-th column with the identity $nh_n = \sum_{r=1}^{n} p_r\, h_{n-r}$. Now use the linearity of the determinant and cyclically move the k-th column to the first column. Observe that those matrices are skew Jacobi-Trudi; Hence, their determinants are the skew Schur functions [8, I Section 5]. ∎

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